Triangle Mysteries

EHRHARD BEHRENDS AND STEVE HUMBLE

This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on. Contributions are most welcome.

The Baltic Centre for Contemporary Art (http://www.balticmill.com) is an international centre for contemporary art located on the south bank of the river Tyne alongside the Gateshead Millennium Bridge in Gateshead, North East England, United Kingdom. It opened in 2002 in a converted flour mill, and presents a constantly changing programme of exhibitions and events.

Steve in action.

To celebrate its tenth anniversary, the Baltic held a weekend of events themed around the number TEN. One of us (S. H.) was asked to present a series of maths art workshops in which the general public was able to participate during this weekend. Turner Prize (for contemporary art) Winner Mark Wallinger's work "Systemising the Randomness of Nature, a look at Super Perfect Numbers", was exhibiting at the Baltic, so randomness and hidden order were chosen as the theme for the workshop. A game was created that families were able to play as a visual starting point to generate discussion around the question of what randomness is. One starts with three different randomly placed colours on a top row of 10. The rule for creating the next row is simply to look at the row above:

- If two consecutive colours in the row above are the same, then place the same colour in between them in the next row.
- If the two consecutive colours in the row above are different, then place the third colour in between them in the next row.

In this way one generates a row of 9, then - with the same rule - a row of 8, and this procedure continues until one arrives at a single colour at the bottom of a triangle.

When the top row was placed, the second author predicted what the final resulting colour at the bottom of the triangle would be.

At first it was not clear that this prediction could be made with one-hundred percent certainty, but after performing this workshop a number of times, it turned out that by looking at the first and last colours in this first row one could predict the bottom colour. One simply has to apply the above rule to the first and last entries of the top row: *This* will be the bottom colour. For example, if these entries are red and blue as in the

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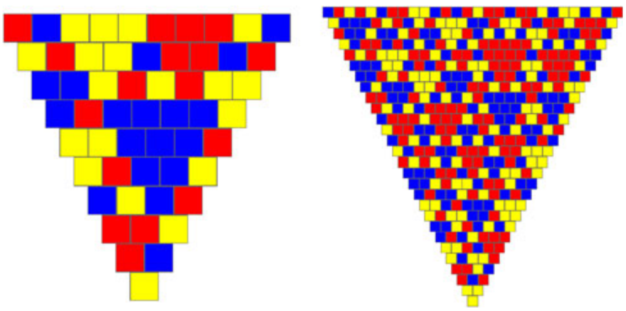
picture below, one can be sure that the final entry at the bottom of the triangle will be yellow.

This provokes the questions:

- Does this only work with a top row of 10?
- Is there some deep underlying order in this randomness?

With the help of a computer program, it turned out that not only 10 but also 4 and 28 are admissible, and this gave rise to the conjecture that the "good" numbers are those of the form $3^s + 1$.



A 10-triangle and a 28-triangle.

In this article, we will investigate this problem in a more general setting. We will consider the rule used for the colours in the Centre as a special case of the following situation:

There are given a finite set Δ with at least two elements (in the gallery example: {red, blue, yellow}) and a map $\phi : \Delta \times \Delta \to \Delta$ (in the gallery: $\phi(i, i) := i$, and if $i \neq j$ then $\phi(i, j) := k$ where $i \neq k \neq j$). The map ϕ induces maps $\phi_n : \Delta^n \to \Delta^{n-1}$ by

$$(a_1, \ldots, a_n) \mapsto (\phi(a_1, a_2), \phi(a_2, a_3), \ldots, \phi(a_{n-1}, a_n)).$$

The $(n-1)$-fold composition $\Phi_n = \phi_2 \circ \ldots \circ \phi_{n-1} \circ \phi_n$ maps Δ^n to Δ.

(Note that the $\phi_n, \phi_{n-1}, \ldots$ generate the rows of a triangle and that $\Phi_n(a_1, \ldots, a_n)$ is the bottom element if the first row is $(a_1, \ldots, a_n)$.)

In the gallery example, the surprising fact was observed that for $n = 4$, 10, 28, the bottom color $\Phi_n(a_1, \ldots, a_n)$ equals $\phi(a_1, a_n)$ for arbitrary $(a_1, \ldots, a_n) \in \Delta^n$.

Call an integer $n > 2$ ϕ-simple if $\Phi_n(a_1, \ldots, a_n) = \phi(a_1, a_n)$ always holds. In the sequel, we will provide some examples and some general results concerning the collection of such ϕ-simple integers. It will turn out that there are some surprising connections with finite abelian groups and arithmetic properties of binomial coefficients.

You, the readers of this article, are invited to transform our results into some entertaining magical tricks, be it by using a deck of cards or simply a sheet of paper in your performance.

AUTHORS

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Examples

We will consider here several classes of examples for which a complete characterization of the ϕ-simple n is possible.

a) *Left or right dependencies*: Let $\psi : \Delta \to \Delta$ be any map, we define $\phi_{\psi,l} : \Delta^2 \to \Delta$ by $(i,j) \to \psi(i)$. Then the following facts are obvious:

- An integer n is $\phi_{\psi,l}$-simple iff ψ^{n-1} coincides with ψ. E.g., for constant ψ all $n > 2$ are $\phi_{\psi,l}$-simple, and it is easy to define ψ such that no $\phi_{\psi,l}$-simple integers exist.
- Suppose that ψ is bijective and let k be the smallest integer such that ψ^k is the identity. Then a number n is $\phi_{\psi,l}$-simple iff n lies in the set $\{k+2, 2k+2, 3k+2, \ldots\}$. For example, if ψ is the identity, all n in 3, 4,... are $\phi_{\psi,l}$-simple, and in the case $\Delta = \{0, \ldots, r-1\}$ and $\psi(i) := i+1 \mod r$ one arrives at the $n \in \{r+2, 2r+2, 3r+2, \ldots\}$.

Similar results hold if ϕ depends only on the right number in the tuple (i, j), that is, if ϕ is defined as $\phi_{\psi,r}(i, j) := \psi(j)$ for a map $\psi : \Delta \to \Delta$.

b) *The case of an abelian group,* I: Suppose that $+ : \Delta \times \Delta \to \Delta$ is such that $(\Delta, +)$ is an abelian group. We consider $\phi^+(i, j) := i + j$. Then, for $n \geq 3$, one has

$$\Phi_n(a_1, \ldots, a_n) = a_1 + \binom{n-1}{1} a_2 + \binom{n-1}{2} a_3 + \cdots + \binom{n-1}{n-2} a_{n-1} + a_n.$$

Consequently, n will be ϕ^+-simple iff for all $k \in \{1, \ldots, n-2\}$ and all $a \in \Delta$ the element $\binom{n-1}{k} a$ is zero.

In order to apply this observation, we need some facts concerning binomial coefficients.

Lemma 1. (i) *Let p be a prime and m an integer such that $m > p$. Then p divides all $\binom{m}{k}$ for $k = 1, \ldots, m-1$ iff m is of the form p^s. In this case p^2 does not divide all these $\binom{m}{k}$.*

(ii) *Let m and r be integers with $m > r > 1$ such that r divides the $\binom{m}{k}$ for $k = 1, \ldots, m-1$. Then r is a prime p, so that by (i) m is of the form p^s.*

Proof. These assertions are a reformulation of a classical result on binomial coefficients due to Balak Ram [3]. (For a far-reaching generalization of Ram's theorem, cf. [2].) It states: The greatest common divisor of the numbers $\binom{m}{k}$, $k = 1, \ldots, m-1$, is p if m is the power p^s of a prime p and it is 1 otherwise.

In the next section, we will derive an independent proof of this fact as a corollary to our results on ϕ-simple numbers (corollary 8).

Proposition 2. *With the notation preceding the lemma, one has:*

(i) *Suppose that p is a prime such that $p \cdot a = 0$ for all $a \in \Delta$. Then an $n \geq 3$ is ϕ^+-simple iff there is an s such that $n = p^s + 1$. If such a p exists, it is uniquely determined.*
(ii) *Suppose that there is no prime p such that all $p \cdot a$ are 0. Then there are no ϕ^+-simple n.*

Proof. The first part of (i) follows from the lemma, the second is a consequence of the fact that $m \cdot a = 0 = n \cdot a$ implies $\gcd\{m, n\} \cdot a = 0$. (ii) Suppose that there exists a ϕ-simple n. Then, by the observation preceding the lemma, the $\binom{n-1}{k} \cdot a$ vanish for $k = 1, \ldots, n-1$ and $a \in \Delta$. Let r be the greatest common divisor of these $\binom{n-1}{k}$. Then all $r \cdot a$ vanish.

The case $r = 1$ can be excluded because we assumed that Δ contains at least two elements. If $r > 0$, however, r must be a prime in contradiction to our assumption.

c) *The case of an abelian group*, II: As in "b" we assume that $(\Delta, +)$ is an abelian group, and this time we define ϕ by $\phi^-(i, j) := -i - j$. We obtain

$$\Phi_n(a_1, \ldots, a_n) = (-1)^{n-1}\left(a_1 + \binom{n-1}{1} a_2 + \binom{n-1}{2} a_3 + \cdots + \binom{n-1}{n-2} a_{n-1} + a_n\right),$$

and we conclude: If an n is ϕ-simple, then: $\binom{n-1}{k} a = 0$ for all $a \in \Delta$ and all $k \in \{1, \ldots, n-2\}$, and $(-1)^{n-1} a = a$ for all a. In particular n must be even if it is not true that $a = -a$ for all a.

Proposition 3. (i) *Suppose that p is a prime such that $p \cdot a = 0$ for all $a \in \Delta$. Then an $n \geq 3$ is ϕ^--simple iff there exists an s such that $n = p^s + 1$.*

(ii) *Suppose that there is no prime p such that all $p \cdot a$ are 0. Then there are no ϕ^--simple n.*

Proof. If p is odd, the proof follows again from the concrete description of the number $\Phi_n(a_1, \ldots, a_n)$. In the case $p = 2$ one has $\phi^+ = \phi^-$, and the result is a consequence of proposition 2.

The Solution of the Mystery: Günter Ziegler has contributed the important observation that the Centre example corresponds to ϕ^- for the group $\mathbb{Z}_3$. Thus it follows from proposition 3 that the ϕ^--simple n in this situation are precisely the numbers $3^s + 1$ as conjectured. (An independent proof is shown below: See the remark after theorem 7.)

We mention that there are some obvious constructions to generate new examples from known ones, and in all of these cases the simple integers for the new maps can easily be determined from the simple integers for the old ones:

- *Products*: Let Δ_i and $\phi_i : \Delta_i \times \Delta_i \to \Delta_i$ for $i = 1,2$ be given. We define $\Delta := \Delta_1 \times \Delta_2$ and $\phi : \Delta \times \Delta \to \Delta$ by $((i_1, i_2), (j_1, j_2)) \mapsto (\phi_1(i_1, j_1), \phi_2(i_2, j_2))$.
- *Projections*: Let $\Delta \subset \Delta'$ and $\tau : \Delta' \to \Delta$ be any map such that $\tau(i) = i$ for $i \in \Delta$: The $i \in \Delta' \setminus \Delta$ are identified with certain elements of Δ. Define $\phi' : \Delta' \times \Delta' \to \Delta'$ by $(i', j') \mapsto \phi(\tau(i'), \tau(j'))$.
- *Permutations*: Let $\tau : \Delta \to \Delta$ be a bijection and $\phi : \Delta \times \Delta \to \Delta$. This situation induces $\phi_\tau(i,j) := \phi(\tau(i), \tau(j))$.

By combining these results, one finds many Δ and ϕ where the ϕ-simple n can be characterized. Even more is true: If the number of elements of Δ is two, resp. three, we have investigated systematically all $2^{(2^2)}$ resp. $3^{(3^3)}$ possible ϕ, and whenever we found a ϕ-simple n, it turned out that this situation could be explained by a suitable combination of the preceding examples and constructions.

Some General Results

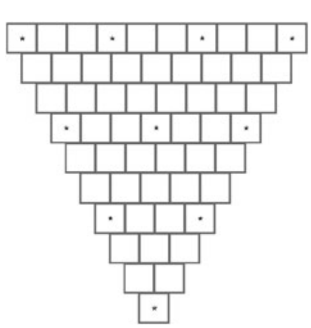
In the next lemma, we show how to obtain new ϕ-simple n from known ones:

Lemma 4. *Let $\phi : \Delta \times \Delta \to \Delta$ be given. If an integer d is ϕ-simple, then so is $(d-1)^s + 1$ for every s.*

Proof. To illustrate the idea. we will consider the case $d = 4$ and $n = 10$ first. We will write the top row $(a_1, \ldots, a_{10})$ as $(a_1^1, \ldots, a_{10}^1)$, the second row $\phi_{10}(a_1, \ldots, a_{10})$ as $(a_1^2, \ldots, a_9^2)$, the third row $\phi_9 \circ \phi_{10}(a_1, \ldots, a_n)$ as $(a_1^3, \ldots, a_8^3)$, etc. The bottom element $\Phi(a_1, \ldots, a_n)$ of the triangle is denoted by a_1^{10}.

We consider six (overlapping) subtriangles. (In the picture, their vertices are indicated by "*".) The first row of the first one is $(a_1^1, \ldots, a_4^1)$. Because 4 is ϕ-simple, we know that a_1^4 equals $\phi(a_1^1, a_4^1)$, and a similar observation applies to the other five triangles.

To phrase it otherwise: In order to calculate all elements in the large triangle, it suffices to consider the triangle formed by the elements with a "*" (first row (a_1^1, a_4^1, a_7^1, a_{10}^1), second row (a_1^4, a_4^4, a_7^4), third row (a_1^7,a_4^7), bottom element a_1^{10}). Because 4 is ϕ-simple, it follows that $a_1^{10} = \phi(a_1^1, a_{10}^1)$, that is, 10 is also ϕ-simple.



The relevant subtriangles

In the general case, one argues in a similar way. The proof is by induction on s, where the assertion is true for $s = 1$ by assumption. Suppose that the result has been shown for a fixed s and that $n = (d-1)^{s+1} + 1$. In the associated triangle, we find $d + (d-1) + \cdots + 2 + 1$ subtriangles where in each one the induction hypothesis can be applied. Thus we may pass to a triangle where the first row contains only d elements, and because d is ϕ-simple, we conclude that n is ϕ-simple as well.

To prepare the next result, we will need a further definition: We will say that a $\phi : \Delta \times \Delta \to \Delta$ is *left and right unique* if all maps $\phi(\cdot, j) : \Delta \to \Delta$ and $\phi(i, \cdot) : \Delta \to \Delta$ are one-to-one. (It follows that i can be reconstructed from j and $\phi(i, j)$, and j from i and $\phi(i, j)$.)

This holds, for example, for the ϕ^+ and the ϕ^- from the preceding section that are induced by abelian groups (parts "b" and "c"). Also in the case of a general finite group $(\Delta, \circ)$, the map $\phi : (i,j) \mapsto i \circ j$ would be left and right unique. However, we do not know how to find ϕ-simple integers in the general nonabelian case.

Note that the examples in subsection "a" of the Example section never are left and right unique.

Proposition 5. *Suppose that $\phi : \Delta \times \Delta \to \Delta$ is left and right unique and that d is a ϕ-simple integer. By r we denote the number of elements in Δ.*

Then an integer $n > d$ such that $n \not\equiv 1 \bmod (d-1)$ will not be ϕ-simple. More precisely, the following is true: If $n \not\equiv 1 \bmod (d-1)$, then the proportion of the $(b_1, \ldots, b_n) \in \Delta^n$ for which $\Phi(b_1, \ldots, b_n) = \phi(b_1, b_n)$ holds is $1/r$ (and not 1 as in the case of ϕ-simple n).

Proof. Let $n > d$ be given with $n \not\equiv 1 \bmod (d-1)$. Consider an arbitrary $(a_1, \ldots, a_{n-d+1}) \in \Delta^{n-d+1}$ in the d'th row of the triangle.

Claim 1 For fixed $i, j \in \Delta$ there are precisely r^{d-3} elements $(b_1, \ldots, b_n) \in \Delta^n$ with $b_1 = i$ and $b_n = j$ such that

$$\phi_{n-d+2} \circ \cdots \circ \phi_n(b_1, \ldots, b_n) = (a_1, \ldots, a_{n-d+1}).$$

Proof of the Claim: In order to explain the idea of the proof, we start with a concrete example. We consider as in the gallery $\Delta = (Z_3, +)$ together with $\phi = \phi^- : \Delta \times \Delta \to \Delta$, $(i,j) \mapsto -i-j$. It was already mentioned that $d = 4$ is ϕ-simple, and we will consider the case $n = 8$, $i = 0$, and $j = 1$.

How many $(0, b_2, \ldots, b_7, 1) \in \Delta^8$ exist such that the fourth row of the triangle coincides with a specific $(a_1, \ldots, a_5)$, for example with (1, 0, 0, 1, 1)?

Because 4 is ϕ-simple, we know that $1 = a_1 = \phi(0, b_4)$, and by uniqueness this implies that $b_4 = 2$. But $1 = a_4 = \phi(b_4, b_7)$, and this enables us to identify b_7 as well: $b_7 = 0$. Similarly we can work backwards: From $1 = a_5 = \phi(b_5, 1)$ we conclude that $b_5 = 1$ and – by using $0 = a_2 = \phi(b_2, b_5)$ – in a next step that $b_2 = 2$. It seems that we are still free to choose b_3 and b_6, but this is not the case: After b_3 is fixed, we can calculate b_6 from the identity $a_3 = \phi(b_3, b_6)$. We have thus shown that there are $3 = r^{d-3}$ admissible choices $(0, b_2, \ldots, b_7, 1)$ to obtain (1, 0, 0, 1, 1) in the fourth row of the triangle.

It remains to repeat this argument for the case of general n, r and d.

Because d is ϕ-simple, we know that $\phi(b_1, b_d) = \phi(i, b_d) = a_1$. By the uniqueness assumption for ϕ, there is precisely one b_d with this property. By the same reason, we know that $\phi(b_d, b_{2d-1}) = a_d$ so that b_{2d-1} is also uniquely determined by i and the $a_1, \ldots, a_{n-d+1}$. In this way we continue to identify b_{3d-2}, b_{4d-3}, etc. We stop at the largest l such that $ld - (l-1) \le n$, and we note that $ld - (l-1) = n$ will not happen since $n \neq 1 \bmod (d-1)$.

Next we work from right to left. We have $\phi(b_{n-d+1}, b_n) = \phi(b_{n-d+1}, j) = a_{n-d+1}$, and this enables us to identify b_{n-d+1}. From $\phi(b_{n-2d+2}, b_{n-d+1}) = a_{n-2d+2}$, we recover b_{n-2d+2}, etc. Note that these positions are different from those when we started from the left: no $ld - (l-1)$ coincides with an $n - l'd + l'$, this follows from $n \neq 1 \bmod(d-1)$.

Summing up, we see that i, j and the $(a_1, \ldots, a_{n-d+1})$ determine many of the b_k. Consider in particular the $k \in \{1, \ldots, d\}$. There is exactly one number $k' := n - l'd + l'$ in this interval where $1 \neq k' \neq d$, and $b_1(= i)$, $b_{k'}$, and b_d are known. Choose any $b_s \in \Delta$ for the $s \in \{2, \ldots, d-1\} \setminus \{k'\}$, and note that there are $d - 3$ such s. After these are fixed, we can determine the remaining b_j. It suffices to work as in the beginning from left to right: b_s together with a_s determines b_{s+d-1}, then we obtain b_{s+2d-2}, etc.

It follows that we have r^{d-3} free parameters, and this proves the claim.

Claim 2 For any $k \in \Delta$, the proportion of $(a_1, \ldots, a_m) \in \Delta^m$ such that $\Phi_m(a_1, \ldots, a_m) = k$ is precisely $1/r$. (This can easily be verified by induction on m; the fact that ϕ satisfies the uniqueness assumption is essential here.)

Now fix any $i, j \in \Delta$ and put $k := \phi(i, j)$. By claim 2 there are r^{n-d+1}/r elements $(a_1, \ldots, a_{n-d+1})$ in Δ^{n-d+1} such that $\Phi_{n-d+1}(a_1, \ldots, a_{n-d+1}) = k$. Each of these $(a_1, \ldots, a_{n-d+1})$ is by claim 1 generated by r^{d-3} elements $(i, b_2, \ldots, b_{n-1}, j)$. In total, there are $r^{d-3} \cdot r^{n-d} = r^{n-3}$ elements $(i, b_2, \ldots, b_{n-1}, j) \in \Delta^n$ that lead to k, and because there are r^{n-2} elements of the form $(i, b_2, \ldots, b_{n-1}, j)$, the proportion is $1/r$ as claimed.

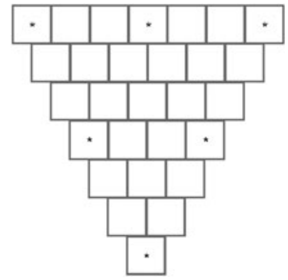
Next we describe what happens in the case $n = 1 \bmod(d-1)$:

Lemma 6. *Suppose that $\phi : \Delta \times \Delta \to \Delta$ is left and right unique and that d is a ϕ-simple integer.*

Let $n > d$ be such that $n = 1 \bmod (d-1)$. Then this number is ϕ-simple iff $(n-1)/(d-1) + 1$ is ϕ-simple.

Proof. The idea is similar to that of the proof of proposition 4. If $d - 1$ is a divisor of $n - 1$, we can reduce the problem of investigating the large triangle with first row $(a_1, \ldots, a_n)$ to the study of a triangle with first row $(a_1, a_d, a_{2d-1}, \ldots)$ of length $(n-1)/(d-1) + 1$.

(As an illustration, consider the example $d = 4$ and $n = 7$ where the reduced triangle consists of the elements marked with a "*" in the picture: There are $(7-1)/(4-1) + 1 = 3$ in the first row.) The result follows since d is ϕ-simple.



From $n = 7$ to $(n-1)/(d-1) + 1 = 3$

Here is our main result:

Theorem 7. *Suppose that $\phi : \Delta \times \Delta \to \Delta$ is left and right unique and that ϕ-simple integers exist. By d we denote the smallest one.*

Then an $n > d$ is ϕ-simple iff n is of the form $(d-1)^s + 1$.

Proof. That all $(d-1)^s + 1$ are ϕ-simple was proved in lemma 4. It remains to show that all ϕ-simple n have this form.

Let such an n be given, we may assume that $n > d$. By proposition 5 we must have $n_1 := n = 1 \bmod (d-1)$. Let $n_2 := (n_1 - 1)/(d-1)$. When n_1 is ϕ-simple, lemma 6 ensures that n_2 is too, so that – again by proposition 2 – $n_2 = 1 \bmod (d-1)$ if $n_2 > d$; this implies $n = 1 \bmod (d-1)^2$. In this way we continue to construct ϕ-simple $n_1 > n_2 > \cdots$ as long as $n_s > d$. In the last step we must have $n_s = d$, hence $n = (d-1)^s + 1$.

Remark: It can easily be checked that 4 is the smallest ϕ-simple number in the gallery example. This leads to another proof of the conjecture that one has to consider, precisely the $3^s + 1$, if one wants to have a similar phenomenon as in the case $n = 10$.

Our results imply the Ram theorem:

Corollary 8. *The greatest common divisor of the $\binom{m}{k}$, $k = 1, \ldots, m-1$, is p if m is of the form p^s for a prime p and 1 otherwise.*

Proof. Let p be a prime. Then, by an elementary argument, p divides all $\binom{p}{k}$ for $k = 1, \ldots, p-1$. This means that $p + 1$ is ϕ^+-simple for $\Delta = \mathbb{Z}_p$ and $\phi^+(i, j) := i + j$ (cf. Examples, part "b"), and p is surely the smallest ϕ^+-simple number. We conclude from theorem 7 that the ϕ^+-simple integers are precisely the $p^s + 1$ so that p will be a divisor of $\binom{p^s}{k}$ for $k = 1, p^s - 1$.

Now let m be such that the greatest common divisor of the $\binom{m}{k}$, $k = 1, \ldots, m-1$ is a number $r > 1$. Choose any prime divisor p of r. Then, with the notation of the preceding paragraph, m is ϕ^+ simple so that, by theorem 7, m is of the form p^s. This can be true for at most one p, and it remains to complete the case $m = p^s$.

We already know that p divides $r := $ the greatest common divisor of the $\binom{p^s}{k}$, $k = 1, \ldots, p^s - 1$ and that p is the only prime divisor of r. Thus $r = p^l$ for some $l \in \mathbb{N}$, and it remains to show that only $l = 1$ can occur. But this can easily be shown: By counting how many p-factors there are in the numerator and the denominator of $\binom{p^s}{p^{s-1}}$, one can convince oneself that $\binom{p^s}{p^{s-1}}$ is not divisible by p^2. This completes the proof.

Some Open Problems

1. All examples of Δ and ϕ where we are able to determine ϕ-simple integers are derived from the Δ and ϕ and the constructions described in the Examples section. It would be interesting to have further classes of examples.
2. Let $\circ : \Delta \times \Delta \to \Delta$ be such that $(\Delta, \circ)$ is a group. Can one find conditions similar to the abelian case such that (with $\phi := \circ$) ϕ-simple integers exist?
3. Suppose that Δ and ϕ are given and that one wants to decide whether there are ϕ-simple n. That a particular n is *not* ϕ-simple can easily be checked: Generate a "large" number of $(a_1 \ldots, a_n) \in \Delta^n$ at random. If n is not ϕ-simple, it is very likely that one finds an example where $\Phi_n(a_1 \ldots, a_n) \neq \phi(a_1, a_n)$. But how many n will have to be tested until one can be sure that ϕ-simple ones do not exist?

It would be desirable to have a result of the following type: For all Δ and ϕ there is an n_0 (that can be easily calculated from Δ and ϕ) such that no ϕ-simple n exist if all $n \leq n_0$ fail to be ϕ-simple.

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