

Mathematics and Music (Mathematical World, Vol. 28)

by David Wright

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It is a commonplace that there are links between the world of mathematics and the world of music. But in the literature on these connections, the two areas play asymmetric roles. The reader is usually assumed to have some mathematical background: Mathematical terms and theories are used with little explanation. These investigations are hardly accessible to nonspecialists.

Mathematics and Music is written in a different spirit. It reviews some basic concepts in both mathematics and music from the very beginning, presuming no background in either of these fields. It's addressed to students of all fields who are interested in both subjects. The 12 chapters cover a wide variety of mathematical and musical themes. Chapter 1 is devoted to "basic concepts." Here, the various sets of numbers are introduced (\mathbb{N} , \mathbb{Q} , etc.), and one also learns, for example, that the integers are well ordered, how to visualize functions by their graphs, and how an equivalence relation is defined. Basics for the musical counterpart include the translation of pitches to notes by the treble and bass clefs, musical intervals (for example, the fifth or the octave), and the use of accidentals. At the end of this chapter, cyclic permutations are introduced to explain how the different modes (Ionian, Dorian, etc.) can be derived from a single scale.

Chapter 2 is concerned with "horizontal structures." How are whole notes, half notes, and so on written, which symbols are used for rests, and how do dots change the length of a note? I never realized before that the length d of a note increases to $d(2 - 1/2^m)$ if the note is m -dotted, a fact proved here by geometric series. It is also explained that translation (resp. transposition, resp. retrogression) of patterns corresponds to replacing $f(x)$ by $f(x - c)$ (resp. $f(x) + c$, resp. $-f(x)$) for functions f .

Let's turn to Chapter 3: Harmony and Related Numerology. The mathematics starts with the algebraic structure of \mathbb{Z}_{12} . In this setting, a major chord is just the sequence (4, 3, 5) of modular intervals. Similarly, diminished chords and many others are described and correctly translated to musical notation. (That is, one must write $E^\#$ and not F in the major chord of $C^\#$.)

Chapter 4 introduces ratios as equivalence classes which one can hear as pitches: The octave, for example, corresponds to 2 : 1. Clearly, the number $\sqrt[12]{2}$ plays an important role here: This is the ratio associated with a semitone. One also learns how intervals can be converted to cents and vice versa.

Logarithms and the exponential function are introduced at the beginning of Chapter 5. The graphs are sketched and

the basic properties summarized. This is used to transform the description of intervals from ratios to a measurement in semitones ("the interval ratio r is measured in semitones by $12 \log_2 r$.")

In Chapter 6, it is noted that the partition of the octave into 12 semitones is rather arbitrary: For every $n \in \mathbb{N}$ one could consider the n -chromatic scale based on the interval ratio $\sqrt[n]{2}$. To be able to discuss this scale in more detail, Euler's ϕ -function is introduced ($\phi(n)$ is the number of integers $k < n$ which are relatively prime to n): There are $\phi(n)$ ways to generate the n -chromatic scale by considering k , $2k$, $3k$, ... (modulo n).

In Chapter 7, the modular arithmetic continues, starting with the properties of \mathbb{N} and ending with a number of basic algebraic notions: monoids, groups, homomorphisms, and so on. It is shown how modular arithmetic can be used to generate a 12-tone row in 12-tone music.

Algebraic investigations are also central to Chapter 8; they culminate in the proof of the fact that \mathbb{Z} is a principal ideal domain. It is noted in passing what prime numbers are and how one can find them with the sieve of Eratosthenes. It is then easy to describe how musical passages are built where patterns of m notes stand against patterns of n notes (with $m \wedge n = 1$).

n -chromatic scales are studied in more detail in Chapter 9. For example, the 19-chromatic scale is appropriate if one wants to generate a scale by an interval of ratio 3.

Calculus is introduced in Chapter 10. The ε - δ definition of continuity is given, and piecewise smooth and periodic functions are defined. As the main result, one learns that such functions have a Fourier expansion. With this background, it is easy to explain the importance of formants for the sound of instruments and the human voice.

Chapter 11 starts with the old observation that two pitches played simultaneously sound "harmonious" when the ratio of their frequencies is rational with small numerator and denominator. It is shown how a scale constructed by using only a just fifth necessarily has a small imperfection, the comma of Pythagoras. And it is proved that it is impossible to avoid irrational numbers in the n -chromatic scales.

Problems concerning tuning are investigated further in Chapter 12. For many centuries, various scales have been proposed: The problem is to have as many justly tuned intervals as possible, and, at the same time, intervals which sound ugly have to be avoided. Advantages and disadvantages of the Pythagorean scale, the mean tone scale and the equal temperament are discussed in some detail in this final chapter.

Each chapter is complemented by (mostly mathematical) exercises of various difficulty, and understanding is facilitated by many graphics and musical scores.

Wright has packed an ambitious overview into 150 printed pages. He had to make choices, and it is legitimate that he followed his own preferences. And, of course, it is a matter of taste which of the many aspects of the theme "mathematics and music" a reader will consider more fascinating, or less.

The author writes that "the treatise is intended to serve as a text for a freshman college course." This purpose is completely achieved. This book can be an inspiring basis

for lectures presented to all students. For readers with a mathematical background who are interested in music, the book will be a valuable resource. And even those who have studied music for many years may learn many new facts (formants, just fifths, equal temperament, to mention a few).

I also recommend these books on the same subject:

- G. Assayag, H. G. Feichtinger, J. F. Rodrigues (eds.): “Mathematics and Music: A Diderot Mathematical forum, Lisbon, Paris and Vienna, December 3–4, 1999” Berlin, Springer, 2002.

- J. Fauvel, R. Floyd, R. Wilson, (eds.): “Music and Mathematics: From Pythagoras to Fractals,” Oxford University Press, Oxford, 2003.
- L. Harkleroad: “The Math behind the Music,” Cambridge University Press, Cambridge and the Mathematical Association of America, Washington D.C., 2007.

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