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# Music and Mathematics

*edited by J. Fauvel, R. Flood, and R. Wilson*

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**REVIEWED BY EHRHARD BEHRENDS**

Many years ago, preparing activities for the general public on the occasion of the ICM'98 in Berlin, the Berlin universities organized several seminars to discuss the relations between mathematics and music. Many different aspects were covered by the lectures, which were given by people working in these fields, among them a number of young composers. For me, it was a surprise to learn that mathematical ideas are rather influential in certain areas of contemporary music (which, however, are not very close to my personal musical interests: classical music and jazz). Also I had not realized before how important physiological facts are to the possibility of using mathematical structures successfully. For example, most of us are unable to recognize the absolute pitch of a tone. Only the relations between different pitches are perceptible: the interval E–G is "the same" as the interval A–C. Also, in many cases a tone may be replaced by its octave without noticeably changing the character of a musical piece. As a consequence, much of the work concerned with scales can be reduced to finding the appropriate pitch ratios between one and two.

"Music and mathematics" is not a well-defined area, but most of what can be said concerns one of the following queries:

1. What are the mathematical principles that underlie the construction of musical scales? What are the defects of the Pythagorean scale, why is $\sqrt[12]{2}$ important?
2. Are there other problems of interest to the "working musician" that can be solved by using (more or less advanced) mathematics? For example, of what help is Fourier analysis for the artisan who wants to construct a guitar?
3. Is mathematics helpful in analyzing musical compositions? Did certain composers have mathematical structures (like symmetry or remarkable relations between numbers associated with the composition) in mind during their work?
4. Can mathematics be used as a toolbox to produce interesting music?

All of these aspects are discussed in this book. The first contribution, "Tuning and temperament" by Neil Bibby, starts with a description of the early attempts of the Pythagoreans to relate harmony to mathematics. They discovered that intervals are considered to sound harmonic if the ratio of the frequencies is a rational number with "small" nominator and denominator. The ratio 2:1 is not very interesting since the octave is somehow identified with the original tone. The Pythagorean scale is based on the ratio 3:2, the *perfect fifth*. If one starts with "C", one first obtains "G", then "D" etc. Here, one has to apply the principle, mentioned earlier, that a note can be identified with its octave. So the fifth of the fifth, the ratio 9:4, is replaced by 9:8 to make the ratio lie between one and two. In this way the notes corresponding to the white keys of the keyboard are generated. (However, to produce the "F" one has to go backwards: "C" is the fifth of "F": "F" is added to the scale for this reason.)

Soon it was realized that many of the intervals which occur in this scale are far from being simple. For example, the frequency ratio of the major third ("C" to "E", say) is 81:64. Also, the Pythagorean scale is not well suited for modulation. If one considers a note already constructed as the keynote of a new scale it will be necessary to include new notes. For example, one has to add "F sharp" when starting with "G". This process never stops. There is no finite scale which is closed under forming Pythagorean scales if one can select any note as a basic key note.

Bibby describes some of the proposals that have been made to overcome these difficulties. For example, we learn how the frequency ratios in the scale of the just intonation are defined and to what extent it is superior to the Pythagorean scale. And that Marin Mersenne designed a keyboard with 31 notes for each octave, which made it possible to distinguish between "F sharp" and "G flat", a distinction which is important in the Pythagorean scale and in the just intonation.

Most readers will know that nowadays nearly all instruments use the equal temperament. The frequency ratio between two adjacent notes of the 12 notes in an octave on a keyboard is always the same, and in this way the twelfth root of 2 comes into play. It is a really "democratic" scale, as each note plays the same role. Nevertheless it is a rather ironical aspect of the history of music that one started with the philosophy of small rational numbers and ended up with a scale where *no* interval (up to the octave) is rational.

But this is not the end of the story. Later (in Chapter 9) we learn why, besides our 12-tone system, certain $n$-tone systems play a role in music theory. Here in particular the cases $n = 13, 19, 21, 31$, and 53 are of some interest. Similar questions are mentioned also in some other contributions. For example, in Chapter 2—a chapter with a more historical than mathematical emphasis—J. V. Field explains how Kepler tried to find musical proportions in various quantities of the solar system.

For me, the most interesting chapters are those of Part Two, "The mathematics of musical sound". First, in Chapter 3, Charles Taylor describes some experiments with real instruments to demonstrate how one can hear combinations of notes. It is a strange fact that the ear sometimes "hears" the difference note of, e.g., 80 Hz if two notes of 500 and 580 Hz are played simultaneously. Taylor has no convincing explanation of this phenomenon; it is argued that the effect is caused by a combination of physical and physiological reasons.

Next, in Chapter 4, Ian Stewart demonstrates that many interesting mathematical problems are touched upon if one wants to calculate the positions of the frets of a guitar correctly. He explains the difference between "construction with circle and ruler" and "construction with circle and *unmarked* ruler" and how simple it is to trisect an angle if one is allowed to mark a distinguished point on the edge of the ruler.

The main part of Stewart's article is the description of Strähle's construction of the position of the frets. Strähle, a Swedish craftsman, suggested his construction in 1743, but it was erroneously argued by Jacob Faggot, of the Swedish Academy, that the argument has a flaw. Strähle had in fact found an approximation of $\sqrt[12]{2}$ by simple geometric means, which in a sense, is optimal: the best approximation of $2^x$ by a function of type $(ax + b)/(cx + d)$ is

$$\frac{(2 - \sqrt{2})x + \sqrt{2}}{(1 - \sqrt{2})x + \sqrt{2}},$$

and Strähle used this function; he approximated $\sqrt{2}$ by 17/12, a ratio which appears when expanding $\sqrt{2}$ as a continued fraction.

The third article in this part (Chapter 5) is David Fowler's essay on Helmholtz. For many readers it will be a surprise to learn that Helmholtz not only was a famous physicist but also a physiologist who worked on the physiological basis of the theory of music. Fowler starts with a description of Helmholtz's experiments with sound generators; they were used to demonstrate combinational tones like the difference notes mentioned earlier. More substantial and mathematically more interesting, however, is the solution Helmholtz proposed for the problem of consonance. A fifth and a fourth, for example, which are defined by the ratios 3:2 and 4:3, are perceived as a harmonious sound, which is more pleasant for the ear than an interval selected at random.

What is the cause of this phenomenon? Some answers, among them those of Plato, Kepler, and Galileo, are sketched (in my opinion, Euler's *gradus suavitatis* should also have been mentioned here).

The starting point of Helmholtz's approach is his *consonance curve*. Imagine two instruments playing a note in unison. If one of the frequencies is slowly increased, one will hear a beating. First it is slow, but it becomes quicker and "more unpleasant". Helmholtz quantified this sensation by associating a "degree of unpleasentness": if the number of beats is $x$, then $\varphi(x)$ measures the "unpleasantness". Obviously one has $\varphi(0) = 0$, and qualitatively it is clear that $\varphi$ will first increase up to an maximum (which is assumed to be around $x = 30$) and then decrease.

In order to have a mathematically simple representation of $\varphi$, Helmholtz chose $\varphi(x) = \lambda x/(30 + x^2)^2$, a choice which of course is somewhat arbitrary. This $\varphi$ is used to explain consonance as follows. If two instruments play an interval, one has to sum up the $\varphi$-values which belong to every pair of frequencies from the list of all pitches which occur in the Fourier expansion of the two notes which constitute the interval.

The result is a rather rough curve with minimum zero at the frequency ratios 1:1 and 2:1. But, remarkably, there are also some steep valleys in the graph at 3:2, 5:4, and the other ratios which correspond to the Pythagorean scale.

Part Three concerns "The mathematical structure of music". I am sure that the fascination one can feel when listening to a Schubert sonata or a Chopin mazurka will never be accessible to a mathematical analysis. There are, however, many "intellectual" aspects of music where a mathematical language can reasonably be applied. For example, group theory naturally comes into play when speaking about musical symmetries: see Chapter 6, "The Geometry of Music", by Wilfrid Hodges. But most of these symmetries are only perceptible by optical inspection of the score. (As an experiment, I suggest playing the notes from the first two bars of a popular song in reverse order. It is rather unlikely that an untrained listener will recognize the original.)

Not only group theory plays a role here. In Chapter 7, in the article by Dermot Roaf and Arthur White on "Bells and mathematics", the emphasis is on combinatorics. "Ringing the changes" is the art of ringing a collection of $n$ bells sequentially such that, at the end, all $n!$ permutations have been heard. In addition, certain conditions must be satisfied, for example, from one round to the next, only transpositions between adjacent bells are admissible. This is so because, otherwise, it would be difficult to perform the sequence with really existing heavy bells. It is interesting to see how this problem can be solved by rather simple algorithms and how the solutions are visualized graphically.

In the last chapter of this part, "Composing with numbers" by Jonathan Cross, we are introduced to some mathematical ideas which have found their way to being used as tools for composers. The story begins with the twelve-tone row of Arnold Schönberg; a number of other examples are also discussed. The idea is always the same. First, one associates certain musical parameters, like pitch or duration, with numbers or more complicated mathematical objects, and then the structure of the mathematical part is translated to a piece of music. For example, one could select a magic square and then use the rows (or the columns, or the diagonals) to define the pitches of the clarinet line or the durations of the bassoon line.

Similar ideas are found in Part Four, "The Composer Speaks" (Carlton Gamer and Robin Wilson on "Microtones and projective planes" and Robert Sherlaw Johnson on "Composing with fractals"). The titles indicate the mathematical source of the compositions. Finite projective planes are used to identify certain subsets of tones. For example, if one wants to select three notes out of seven in such a way that the selection generates a "cyclic design", one finds everything that is needed in the geometry of the Fano plane. (A cyclic design in this case is a pattern such that translations modulo 7 give rise to subsets of $\{0, \ldots, 6\}$ in which each pair of numbers is contained in precisely one of the translations.)

Dynamical systems are very common in contemporary music. Here the well-known two-dimensional iterative patterns which lead to the Mandelbrot set generate the musical material. For example, if the channel of the synthesizer has to be determined where the next note will be generated, then a discretization of the $y$-value of the present position of the system is important: e.g., if it lies in $[\ 7,8\ [$, then choose channel 5.

It should be noted that the book is very carefully edited. It is a pleasure to read, and there are many interesting pictures and scores to illustrate the material. Readers who are particularly interested in the historical part of the subject can consult the book *Mathematics and Music* (edited by Gerard Assayag, Hans-Georg Feichtinger, and Jose Rodriguez, Springer 2002; reviewed in *The Mathematical Intelligencer*, vol. 27, no. 3, p. 69). There is, surprisingly, only a small overlap in the content of these two books. The generation of scales by mathematical principles naturally plays a prominent role in both of them.

For me, only two aspects are missing. The first omission: I would have appreciated an article on Euler's work on music. He was one of the first to relate mathematics to consonance, and it would be interesting to compare his work with that of Helmholtz. And I was surprised to see that one cannot find anything substantial on "probability and music". In the music of the last century there is an abundance of examples in which the building blocks of certain compositions are generated stochastically, be it the pitches, the durations, or even the wave forms of the sounds.

But these objections are not essential. Let's praise the editors that they have presented an attractive volume that covers almost all of the important aspects of the interplay between mathematics and music.

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